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Finite Element Method to the Solution of Forcing Term in a Non-homogeneous Equation of Statics

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Abstract
Finite element method is a numerical means of finding an approximate solution to a partial differential equation. It's applications span to almost every area of life, from heat transfer to practical engineering problems. A concrete beam was discretized into nth thousand mesh of 0.5 millimetres. A complex variable function method alongside Green's theorem were used to transform the discretized beam into a nonhomogeneous governing equation of statics. Thereafter, finite element method was used to determine the forcing term by first finding the weak form of the governing equation. The solution revealed a linear relationship between force and displacement, confirming Hooke's law. That is, increase in displacement leads to the increase in the forcing component.

Keywords: Finite Element, Discretization, Mesh, Physical Parameter, Forcing Equation

1. Introduction

Finite element analysis is a very dependable mathematical tool for solving engineering problems that are either simple in nature or complex in configuration. Finite Element Method (FEM) was first adopted in 1956 for aircraft structural problem analysis. After then, within ten years interval, the credibility of the method for the solution of different types of applied science and engineering problem were identified (Rao, 1982) [8].

Finite element method has demonstrated to be among the very reliable techniques for analyzing the efficiency of a wide range of practical engineering problems. It is also widely used in applied mathematics. The key concept in the finite element method is actually to find the solution of a complicated problem by exchanging it with a simpler one (Rao, 1992) [9].

Currently, the finite element method includes the use of piecewise continuous domain. The FEM was first initiated by Courant in 1943 in a literature of applied mathematics (Courant, 1943) [4]. Courant's research work was overlooked until engineers developed a new format for analyzing stress, which became a good substitute for the analyses of elastic bars of regular configuration (size). Properties of the bar were determined in such we can approximate joint displacements of points in a continuum (Darrel *et al*., 2006) [5].

We must recognize that the advancement of finite element method was motivated by the advances in computer architecture and software development.

Mathematicians and Engineers have developed different techniques to discretize continuum mechanics problems. Some of the methods are the finite difference method (FDM) and the residual approach which are dependable in the determination of extreme values of some functional. Also, some engineers have made several attempts to solve the problem from a physical point of view. (McHenry, 1943) and (Newmark, 1949) [7]. These methods revealed that it was possible to analyze a continuum by detaching a fraction of the body with an arrangement of trusses. Argyris (1650 and 1954) [1] and Turner *et al*. (1956) conjucture that a more direct replacement of properties could be achieved. These findings were anchored upon basic hypothesis of the behaviour of elements. This realization caused a closer tide between the difference based approach and the pure mathematical approach, an agreement was reached which was the foundation that further enhances the development of finite element method. For several years, finite element method was not considered to be numerical, and mathematically well defined, discretization technique for simulating and analyzing a wide range of boundary value problems.

Finite element method is a very vast technique, and has found applications in several engineering problems. However, it has not been used to solve problem involving the non-homogeneous equation of statics in the theory of elastic mixture. The simplification of this complex analytic equation into its numerical linear algebraic equation, is what makes this article unique.

2. Mathematical Formulation

A cross-sectional area A, of a concrete beam is discretized into nth-thousand mesh of 0.5 squared millimetres each. Suppose the wall is subjected to concentrated load q, body forces (F_B), and other forces such as wind load; that which is, capable of initiating change in solids. As shown in Figure 1. Suppose deformation of the beam is determined by the variation in its primary field variables. Hence, we are to determine the forcing equation, and its displacement field ψ, when other parameters remain constant.

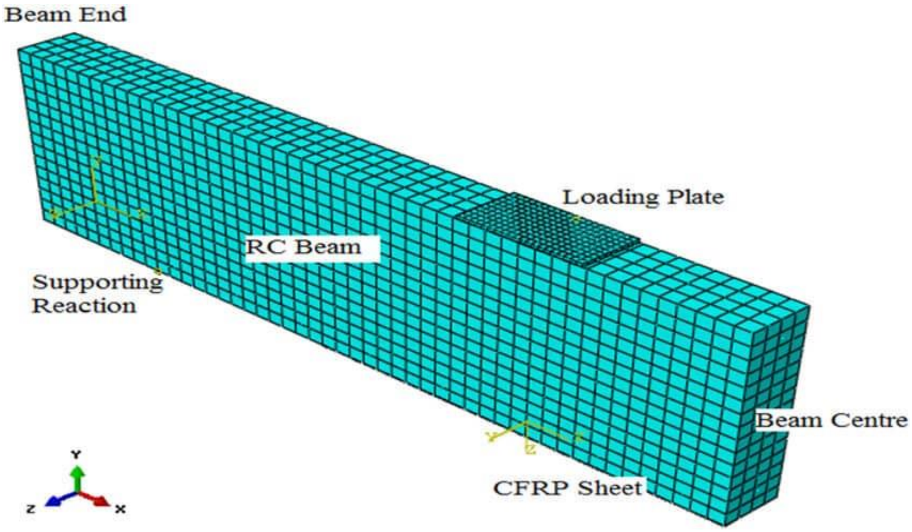

Fig 1: Discretized Concrete Beam under External Force

3. Solution Method

Adopting the formulation by (Basheleishvili, 1997)

$$a\Delta u + c\Delta u + b\nabla(\nabla . u) + d\nabla(\nabla . u) = w \qquad (1)$$

Where Δ is two dimensional Laplacian, ∇ is the principal operator of field theory, u is the displacement vector and a, b, c, and dare constitutive constants characterizing the physical behaviour of the elastic mixture, while w represents the transformed state of the mixture defined as,

$w = u + iv$

3.1. Complex Variable Function

Simplifying equation (1), using complex variable as follows:

$$z = x + iy \qquad (2)$$

And

$$\bar{z} = x - iy \qquad (3)$$

Adding equations (2) and (3)

$$2x = z + \bar{z} \qquad (4)$$

Introducing partial differential operator to (4)

$$2\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \qquad (5)$$

Subtracting equation (3) from (2)

$$2iy = z - \bar{z} \qquad (6)$$

Expressing (6) as partial differential equation

$$2\mathrm{i}\frac{\partial}{\partial y} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \tag{7}$$

Adding equation (5) and (7); we have,

$$2\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right) = 2\frac{\partial}{\partial z} \tag{8}$$

Subtracting equation (5) from (7); that is,

$$2\left(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\right) = 2\frac{\partial}{\partial \bar{z}} \tag{9}$$

Multiplying equations (8) and (9); we have,

$$4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 4i\left(\frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial x \partial y}\right) \tag{10}$$

Replacing the right hand part of equation (10) with (5) and (7) respectively; we get,

$$4\frac{\partial^2}{\partial z \partial \bar{z}} = 4\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right) - 4i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \tag{11}$$

Expressing the displacement component u in its complex form as:

$$u_1 + iu_2 = \varphi \tag{12}$$

And its conjugate as,

$$u_1 - iu_2 = \bar{\varphi} \tag{13}$$

Substituting (12) and (13) into equation (10) as components of the dependent variable.

That is,

$$4\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} = 4\left(\frac{\partial \varphi}{\partial z} + \frac{\partial \bar{\varphi}}{\partial \bar{z}}\right) - 4i\left(\frac{\partial \varphi}{\partial z} - \frac{\partial \bar{\varphi}}{\partial \bar{z}}\right) \tag{14}$$

To simplify equation (1) we shall let

$$\Delta \mathrm{u} = 4\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \tag{15}$$

$$\frac{\partial \varphi}{\partial \mathrm{z}} + \frac{\partial \bar{\varphi}}{\partial \bar{\mathrm{z}}} = 2\nabla.\mathrm{u} = 2\theta \tag{16}$$

Substituting equation (15) and (16) in equation (1); we get,

$$4\mathrm{a}\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} + 4\mathrm{c}\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} + 2b\nabla\theta + 2d\nabla\theta = w \tag{17}$$

The Laplacian is defined as:

$$\Delta = \nabla.\nabla = \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \tag{18}$$

Such that

$$\nabla = \frac{\partial}{\partial z} \tag{19}$$

Substituting equation (19) into (17); we have,

$$4\mathrm{a}\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} + 4\mathrm{c}\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} + 2b\frac{\partial \theta}{\partial z} + 2d\frac{\partial \theta}{\partial z} = w \tag{20}$$

$$\frac{\partial}{\partial z}(4\text{a}\frac{\partial \varphi}{\partial \bar{z}} + 4\text{c}\frac{\partial \varphi}{\partial \bar{z}} + 2b\theta + 2d\theta) = w \tag{21}$$

$$\int d\left(4\text{a}\frac{\partial \varphi}{\partial \bar{z}} + 4\text{c}\frac{\partial \varphi}{\partial \bar{z}} + 2b\theta + 2d\theta\right) = \int w\, dz \tag{22}$$

$$4\text{a}\frac{\partial \varphi}{\partial \bar{z}} + 4\text{c}\frac{\partial \varphi}{\partial \bar{z}} + 2b\theta + 2d\theta = \oint w\, dz \tag{23}$$

3.2. Transformation of the Non-Homogeneous Part

$$\oint w\, dz \tag{24}$$

Where

$$\text{w} = \text{u} + \text{iv and z} = \text{x} + \text{iy}$$

Then, equation (24) becomes,

$$\int (u + iv)d(x + iy) = \int (u + iv)\,(dx + idy)$$

$$= \int (udx - vdy) + i\int (vdx + udy) \tag{25}$$

Applying Green’s Theorem to the line integral in equation (25), we have an area integral of the form (Dass, 2012)

$$\iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \tag{26}$$

Equating like terms of equation (25) and (26); we have,

$$\iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy \; = \int (udx - vdy) \tag{27}$$

And

$$\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = \int (vdx + udy) \tag{28}$$

From Cauchy Morare Theorem, the line integrals in equation (27) and (28) is equated to zero; leaving,

$$\iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = 0 \;\Rightarrow \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{29}$$

And

$$\iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy = 0 \;\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{30}$$

Equation (29) and (30) satisfy Cauchy Riemman conditions. The necesssary condition for the transform (w) to be analytic (Murray, 2009).
Then, the sufficient condition is obtain by differentiating equation (29) and (30) respectively with respect to y and x. That is,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0 \tag{31}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} = 0 \tag{32}$$

Equating equation (31) and (32); we have,

$$\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{33}$$

Hence, equation (33) is the governing differential equation representing Figure 1.

3.3. Solution of the Forcing Term by Finite Element Method

The weak form of equation (33) is given as,

$$\mathrm{AE}\iint \mathrm{W}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)\mathrm{dA}=0 \tag{34}$$

Integrating by part, the first term in the bracket; we have,

$$\int_{x_l}^{x_u}\mathrm{W}\frac{d^2\psi}{dx^2}dx=\left(\mathrm{W}\frac{d\psi}{dx}\right)_{x_l}^{x_u}-\int_{x_l}^{x_u}\frac{dW}{dx}\frac{d\psi}{dx}dx \tag{35}$$

Similarly, we write (using the Green Gauss Theorem) (Dass, 2012).

$$\iint \mathrm{W}\frac{\partial^2\psi}{\partial x^2}\mathrm{dA}=\int_{y_l}^{y_u}\left(\mathrm{W}\frac{\partial\psi}{\partial x}\right)_{x_l}^{x_u}dy-\iint\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}\mathrm{dA} \tag{36}$$

The general interpretation of x_l, x_u, and y_l, y_u are presented in (Zuonaki and Ebikiton, 2024). Therefore,

$$dy=\pm l_x ds \tag{37}$$

Where l_x is the direction cosine of the out-ward normal $\vec{n}$, **and** ± is introduced to take care of both ends.

$$\int_{y_l}^{y_u}\left(\mathrm{W}\frac{\partial\psi}{\partial x}\right)_{x_l}^{x_u}dy=\oint \mathrm{W}\frac{\partial\psi}{\partial x}l_x ds \tag{38}$$

Replacing the second term of equation (37) with equation (38); we get,

$$\iint \mathrm{W}\frac{\partial^2\psi}{\partial x^2}d=-\iint\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}\mathrm{dA}+\oint \mathrm{W}\frac{\partial\psi}{\partial x}l_x ds \tag{39}$$

By a similar procedure; we have,

$$\iint \mathrm{W}\frac{\partial^2\psi}{\partial y^2}d=-\iint\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}\mathrm{dA}+\oint \mathrm{W}\frac{\partial\psi}{\partial y}l_y ds \tag{40}$$

Hence

$$\mathrm{AE}=\mathrm{AE}\left[-\iint\left(\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}\right)\mathrm{dA}+\oint \mathrm{W}\frac{\partial\psi}{\partial x}l_x ds+\oint \mathrm{W}\frac{\partial\psi}{\partial y}l_y ds\right]=0$$

Therefore

$$\mathrm{AE}\iint\left(\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}\right)\mathrm{dA}=\mathrm{AE}\oint \mathrm{W}\left(\frac{\partial\psi}{\partial x}l_x+\frac{\partial\psi}{\partial y}l_y\right)ds=\mathrm{AE}\oint \mathrm{W}\left(\frac{\partial\psi}{\partial n}\right)ds$$

$$\mathrm{AE}\iint\left(\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}\right)\mathrm{dA}=\mathrm{AE}\oint \mathrm{W}\left(\frac{\partial\psi}{\partial n}\right)ds \tag{41}$$

$$\left(\mathrm{AE}\iint[B]^T[B]\mathrm{dA}\right)\{\psi\}=\mathrm{AE}\oint[N]^T\left(\frac{\partial\psi}{\partial n}\right)ds \tag{42}$$

$$[k]\{\psi\}=\{f\} \tag{43}$$

Where the stiffness matrix,

$$[k]=\mathrm{AE}\iint[B]^T[B]\mathrm{dA}$$

And the loading force,

$$\{f\}=\mathrm{AE}\oint[N]^T\left(\frac{\partial\psi}{\partial n}\right)\mathbf{ds}$$

Then, for the entire finite element mesh; we have,

$$\mathrm{AE}\sum_1^{NELEM}\left(\iint[B]^T[B]\mathrm{dA}\right)\{\psi\}=\mathrm{AE}\sum_1^{NELEM}\oint[N]^T\left(\frac{\partial\psi}{\partial n}\right)ds \tag{44}$$

Where [N] and $[N]^T$ are the shape function matrix and its transpose respectively. [B] and $[B]^T$ are the derivatives of the shape function, while $\{\psi\}$ is the displacement vector.
Note: In the Galerkin formulation, we use the same shape function as the weight function ($\mathbf{W} = \mathbf{N}$) (Shehu; 2012) [10].

3.3.6. Computation of Interpolation Function

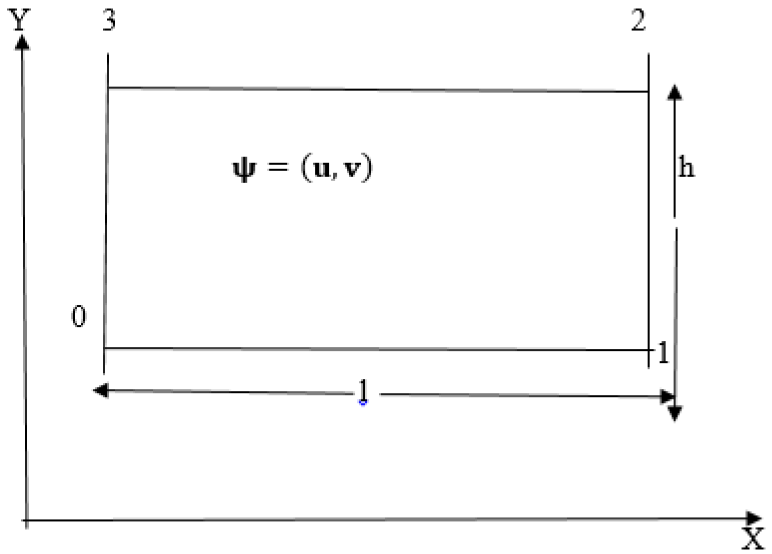

Fig 2: Displacement Distribution in Rectangular Mesh

Let the axial and transverse displacements in the quadrilateral be represented by **u and v** respectively.
Here, the unknown field variables vary independently along two directions. Hence, we assume that the displacement field over the mesh is given as:

$$u(x, y) = C_0 + C_1x + C_2y + C_3xy \tag{45}$$

And

$$v(x, y) = C_0 + C_1x + C_2y + C_3xy \tag{46}$$

Thus, for the rectangular element of size$(l \times h)$. On computation; we have,

$$u_1 = C_0 \tag{47}$$

$$u_2 = C_0 + C_1l \tag{48}$$

$$u_3 = C_0 + C_1l + C_2h + C_3lh \tag{49}$$

$$u_4 = C_0 + C_4h \tag{50}$$

Solving for, $C_o, C_1,\ C_2,$ and C_3; we have,

$$C_0 = u_1 \tag{51}$$

$$C_1 = \frac{u_2 - u_1}{l} \tag{52}$$

$$C_2 = \frac{u_4 - u_1}{h} \tag{53}$$

$$C_3 = \frac{u_3 + u_1 - u_2 - u_4}{lh} \tag{54}$$

Substituting for, $C_0, C_1,\ C_2,$ and C_3; in equation (45); we have,

$$u(x, y) = u_1 + \left(\frac{u_2 - u_1}{l}\right)x + \left(\frac{u_4 - u_1}{h}\right)y + \left(\frac{u_3 + u_1 - u_2 - u_4}{lh}\right)xy \tag{55}$$

Expanding and grouping like terms; we get,

$$u(x,y) = \left(1 - \frac{x}{l} - \frac{y}{h} + \frac{xy}{lh}\right) u_1 + \left(\frac{x}{l} - \frac{xy}{lh}\right) u_2 + \left(\frac{xy}{lh}\right) u_3 + \left(\frac{y}{h} - \frac{xy}{lh}\right) u_4 \qquad (56)$$

In standard finite element notation equation (56) is written as:

$$u(x,y) = [N_1, N_2, N_3, N_4] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \qquad (57)$$

By a similar procedure; we have

$$,v(x,y) = [N_1, N_2, N_3, N_4] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} \qquad (58)$$

Where

$$N_1 = 1 - \frac{x}{l} - \frac{y}{h} + \frac{xy}{lh}$$
$$N_2 = \frac{x}{l} - \frac{xy}{lh}$$
$$N_3 = \frac{xy}{lh}$$
$$N_4 = \frac{y}{h} - \frac{xy}{lh}$$

Hence, N_i is the shape function, describing the uniform distribution of the displacement field. Since Figure 1; is used to model structural mechanics problems, each node will have two degrees of freedom viz: u and v; hence, we can write the displacement field, using the shape functions derived in equation (57) and (58) as:

$$[N_i]\{\psi\} = [N_i] \begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \qquad (59)$$

And

$$\begin{bmatrix} \frac{dN_i}{dx} \\ \frac{dN_i}{dy} \end{bmatrix} = [B] = \begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} & 0 & \frac{dN_3}{dx} & 0 & \frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_1}{dy} & 0 & \frac{dN_2}{dy} & 0 & \frac{dN_3}{dy} & 0 & \frac{dN_4}{dy} \end{bmatrix}$$

Then;

$$[B]^T[B] = \begin{bmatrix} \frac{dN_1}{dx} & 0 \\ 0 & \frac{dN_1}{dy} \\ \frac{dN_2}{dx} & 0 \\ 0 & \frac{dN_2}{dy} \\ \frac{dN_3}{dx} & 0 \\ 0 & \frac{dN_3}{dy} \\ \frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_4}{dy} \end{bmatrix} \begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} & 0 & \frac{dN_3}{dx} & 0 & \frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_1}{dy} & 0 & \frac{dN_2}{dy} & 0 & \frac{dN_3}{dy} & 0 & \frac{dN_4}{dy} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{dN_1}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_1}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_4}{dy} \\ \frac{dN_2}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_2}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_4}{dy} \\ \frac{dN_3}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_3}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_4}{dy} \\ \frac{dN_4}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_4}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_4}{dy} \end{bmatrix} \quad (60)$$

$$\frac{AE}{l}\begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.2 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.2l & 0 & 0.2 & 0 & -0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2l & 0 & 0.3 & 0 & -0.3l & 0 & -0.2l \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & -0.3l & 0 & 0.3 & 0 & 0.2 \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & -0.2l & 0 & 0.2 & 0 & 0.3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = l\rho g\begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + q\begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \quad (61)$$

Where, the first term in equation (61) represents the stiffness matrix of the solid, the second term represents the axial and transverse displacement vector, the third term stands for body forces, and the fourth term denotes external force. Hence, equation (61) is the numerical version of the forcing equation.

Where

$$KU = F_C + F_B = F$$

$$F = KU \quad (62)$$

Taking the stiffness constant K=2, in equation (62), we generate the following table

Table 1

F	2	4	6	8	10
U	1	2	3	4	5

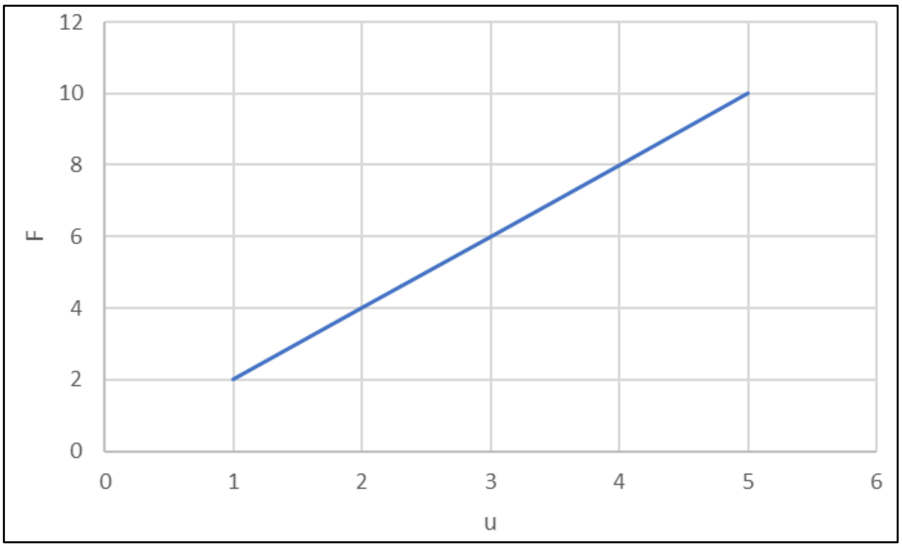

Fig 3: Graph of Force and Displacement

The above shows a linear relationship between forcing term F and displacement U

4. Discussion of Result

The graph above shows the linear relationship between the two physical quantities, force and displacement. Confirming Hooke’s law that states, displacement is directly proportional to the applied force. That is, increase in displacement will lead to the increase in forcing components.

5. Conclusion

The non-homogeneous equation of statics represented by Figure 1, is an analytic problem that is difficult to simplify, inorder to ascertain the field parameters contain in the geometry of a beam under external force. FEM resolve this problem by discretizing the domain and linearizing the non-homogeneous equation into an algebraic equation that is easy to solve.

Finite element method is a numerical means of finding an approximate solution to a partial differential equation. It obtain its power for problem solving from the discretization of any complex geometry into finite element mesh; then, the physical parameters of the problem is determined at that unit level, and the unit result is used to assemble the total number of mesh the geometry is discretized into. The mathematics of this process entails the convertion a governing partial differential equation which represent the geometry into a system of linear equations that is consistent with Hooke’s law.

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